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INTERACTION OF CRACKS WITH BONDS BETWEEN THEIR FACES IN AN ISOTROPIC MEDIUM REINFORCED BY A REGULAR SYSTEM OF STRINGERS

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A problem of an elastic isotropic medium with a system of foreign (transverse with respect to crack alignment) rectilinear inclusions is considered. The medium is assumed to be attenuated by a periodic system of rectilinear cracks with zones where the crack faces interact with each other. These zones are assumed to be adjacent to the crack tips, and their sizes can be commensurable with the crack size. Interaction between the crack faces in the tip zone is modeled by introducing bonds (adhesion forces) between the cracks with a specified strain diagram. The boundary-value problem of the equilibrium of a periodic system of cracks with bonds between their faces under the action of external tensile loads and forces in the bonds is reduced to a nonlinear singular integrodifferential equation with a kernel of the Cauchy kernel type. The condition of critical equilibrium of the cracks with the tip zones is formulated with allowance for the criterion of critical tension of the bonds. A case of a stress state of the medium containing zones where the crack faces interact with each other is considered.

Key words: *periodic system of cracks, stringers, isotropic medium, adhesion forces, contact stresses.*

Deformation of a plate reinforced by a regular system of ribs whose cross sections are narrow rectangles was considered in many publications (see, e.g., [1–4]). Much attention was paid to studying the fracture of a plate reinforced by a regular system of stringers [5–9]. An isolated Griffith crack was considered in all those papers.

1. Formulation of the Problem. An elastic isotropic medium with a system of foreign transverse rectilinear inclusions is considered. Such a medium can be interpreted as an unbounded plate reinforced by a regular system of ribs whose cross sections are narrow rectangles. At infinity, the reinforced plate is subjected to homogeneous tension along the stringers with a stress $\sigma_y^\infty = \sigma_0$ (Fig. 1). The hypothesis assumed for the stringer implies that the stringer thickness remains unchanged during its deformation, and the stress state is uniaxial. The stringers are not subjected to bending and experience tensile forces only.

The stringers are assumed to be attached at discrete points $z = \pm(2m+1)L \pm iky_0$ ($m = 0, 1, 2, \ldots$ and $k = 1, 2, \ldots$) with a constant step over the entire length of the stringer aligned symmetrically with respect to the plate surface. The Cartesian coordinate system and notation used are shown in Fig. 1.

This reinforced medium is assumed to be attenuated by a periodic system of rectilinear cracks. The crack model with zones of interaction of the crack faces is considered. These zones are assumed to be adjacent to the crack tip, and their sizes can be commensurable with the crack size. The adhesion forces continuously distributed in the tip zone are not known in advance and have to be determined in the course of solving the problem. The action of the stringers in the computational scheme is modeled by unknown equivalent concentrated forces applied to the points of junction between the ribs and the medium. The values of the concentrated forces have to be determined in the course of solving the problem. Interaction between the crack faces in the tip zones is modeled by introducing bonds (adhesion forces) between the crack faces with a specified strain diagram. External tensile stresses σ_0 and concentrated forces P_{mn} ($m = \pm1, \pm2, \ldots$; $n = \pm1, \pm2, \ldots$) induce the forces $q(x)$ in the bonds between the crack

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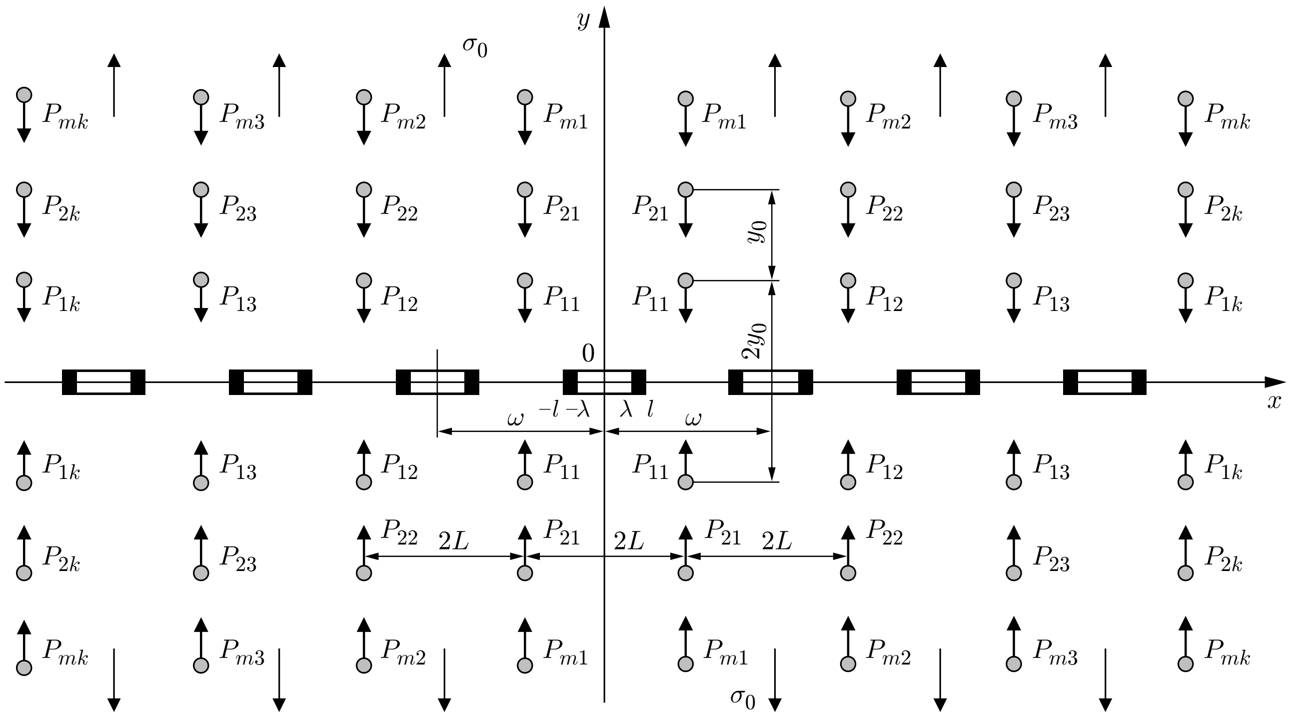

Fig. 1. Computational scheme of the problem.

faces, which have only the normal component by virtue of the problem symmetry. The value of these forces $q(x)$ is not known in advance. Let us choose segments of length $d = l - \lambda$ (tip zone; $y = 0$; $\lambda \leqslant |x - m\omega| \leqslant l$) adjacent to the crack tip, where the crack faces interact with each other. The physical nature of such bonds and the size of the tip zone with interaction of the crack faces depend on the material properties [10–12].

As the sizes of the tip zones of the cracks are small as compared with the plate size, these zones can be conceptually removed and replaced by cuts whose surfaces interact in accordance with a certain law corresponding to the action of the removed material.

In the problem considered, the boundary conditions have the form

$$\sigma_y - i\tau_{xy} = \begin{cases} 0, & y = 0, \quad |x - m\omega| < \lambda, \\ q(x), & y = 0, \quad \lambda \leqslant |x - m\omega| \leqslant l. \end{cases} \tag{1.1}$$

The constitutive relations of the problem posed should be supplemented with an equation relating the displacements of the crack faces and the forces in the bonds. Without loss of generality, this equation can be presented as [13, 14]

$$v^+(x) - v^-(x) = C(x, q)q(x), \qquad \lambda \leqslant |x - m\omega| \leqslant l, \tag{1.2}$$

where $C(x, q)$ is the effective compliance of the bonds, depending on their tension.

With the use of the Kolosov–Muskhelishvili formulas [15] and the boundary conditions (1.1) on the crack faces, the problem solution is reduced to determining two analytical functions $\Phi(z)$ and $\Psi(z)$ from the boundary conditions

$$\Phi(x) + \overline{\Phi(x)} + x\Phi'(x) + \Psi(x) = \begin{cases} 0, & |x - m\omega| < \lambda, \\ q(x), & \lambda \leqslant |x - m\omega| \leqslant l. \end{cases} \tag{1.3}$$

2. Solution of the Boundary-Value Problem. The solution of the boundary-value problem (1.3) is sought in the form

$$\varphi(z) = \varphi_0(z) + \varphi_1(z), \qquad \psi(z) = \psi_0(z) + \psi_1(z) \tag{2.1}$$

with $\Phi(z) = \varphi'(z)$ and $\Psi(z) = \psi'(z)$; the functions $\varphi_0(z)$ and $\psi_0(z)$ determine the stress and strain fields in the reinforced medium without cracks.

In the case considered, the functions $\varphi_0(z)$ and $\psi_0(z)$ are taken in the form

$$\varphi_0(z) = \frac{1}{4}\,\sigma_0 z - \frac{i}{2\pi(1+\varkappa_0)h} \sum_{m=-\infty}^{\infty}{}' \sum_{n=-\infty}^{\infty}{}' P_{mn} \ln \frac{z - m_* L + i y_0 n}{z - m_* L - i y_0 n},$$

$$\psi_0(z) = \frac{1}{2}\,\sigma_0 z - \frac{i\varkappa_0}{2\pi(1+\varkappa_0)h} \sum_{m=-\infty}^{\infty}{}' \sum_{n=-\infty}^{\infty}{}' P_{mn} \ln \frac{z - m_* L + i y_0 n}{z - m_* L - i y_0 n} \tag{2.2}$$

$$- \frac{i}{2\pi(1+\varkappa_0)h} \sum_{m=-\infty}^{\infty}{}' \sum_{n=-\infty}^{\infty}{}' P_{mn} \Big(\frac{m_* L - i y_0 n}{z - m_* L - i y_0 n} - \frac{m_* L + i y_0 n}{z - m_* L + i y_0 n} \Big).$$

Here, the primed sum means that the index $n = m = 0$ is eliminated during summation, h is the plate thickness, and $\varkappa_0$ is the Muskhelishvili elastic constant. To determine the analytical functions $\Phi_1(z)$ and $\Omega_1(z) = z\Phi_1'(z) + \Psi_1(z)$, we use Eqs. (2.1) and (1.3) to obtain the boundary-value problem

$$\Phi_1(x) + \overline{\Phi_1(x)} + \Omega_1(x) = \begin{cases} f(x), & y = 0, \quad |x - m\omega| < \lambda, \\ f(x) + q(x), & y = 0, \quad \lambda \leqslant |x - m\omega| \leqslant l, \end{cases} \tag{2.3}$$

where $f(x) = -[\Phi_0(x) + \overline{\Phi_0(x)} + x\Phi_0'(x) + \Psi_0(x)]$.

As the problem is symmetric with respect to the x axis, the function $f(x)$ is real; therefore, we have $\operatorname{Im}\Omega_1(z) = 0$ on the entire real axis, as is seen from Eq. (2.3). Therefore, taking into account the conditions at infinity, we obtain $\Omega_1(z) = 0$.

Thus, we obtain the Dirichlet problem for the function $\Phi_1(z)$:

$$\operatorname{Re}\Phi_1(z) = \begin{cases} f(x)/2, & y = 0, \quad |x - m\omega| < \lambda, \\ (f(x) + q(x))/2, & y = 0, \quad \lambda \leqslant |x - m\omega| \leqslant l, \end{cases}$$

$$\Phi_1(z) \to 0 \quad \text{as} \quad z \to \infty. \tag{2.4}$$

The sought solution of problem (2.4) is written in the form

$$\Phi_1(z) = \frac{1}{2\pi i X(z)} \int_{-l_0}^{l_0} \frac{F_*(x)(\pi/\omega)\cos(\pi x/\omega)}{\sin(\pi x/\omega) - \sin(\pi z/\omega)}\,dx. \tag{2.5}$$

Here, we have

$$X(z) = \sqrt{\sin^2(\pi z/\omega) - \sin^2(\pi l/\omega)}, \qquad l_0 = \sin(\pi l/\omega),$$

$$F_*(x) = \begin{cases} f(x)X(x), & |x - m\omega| < \lambda, \\ [f(x) + q(x)]X(x), & \lambda \leqslant |x - m\omega| \leqslant l \end{cases}$$

[$X(z)$ is the branch of the function that has the form $\sin(\pi z/\omega)$ at high values of z ($|z| \to \infty$)].

To determine the potential $\Phi(z)$, we have to find the forces in the bonds $q(x)$ in the tip zones of the cracks, i.e., at $\lambda \leqslant |x - m\omega| \leqslant l$, and also the concentrated forces P_{mn} ($m = \pm 1, \pm 2, \ldots$, $n = \pm 1, \pm 2, \ldots$), which are involved into the formulas for $f(x)$, $\Phi_0(z)$, and $\Psi_0(z)$.

3. Determination of Forces in the Bonds. Using the Kolosov–Muskhelishvili formulas and the boundary values of the function $\Phi_1(z)$, we obtain the following equality on the segments $|x - m\omega| \leqslant l$:

$$\Phi_1^+(x) - \Phi_1^-(x) = \frac{2\mu i}{1+\varkappa_0}\,\frac{\partial}{\partial x}\,(v^+ - v^-). \tag{3.1}$$

Using the Sokhotsky–Plemelj formulas [15] and taking into account Eq. (2.5), we find

$$\Phi_1^+(x) - \Phi_1^-(x) = -\frac{i}{\pi X_*(x)} \int_{-l_0}^{l_0} \frac{X_*(t)[f(t)+q(t)](\pi/\omega)\cos(\pi t/\omega)}{\sin(\pi t/\omega) - \sin(\pi x/\omega)}\, dt, \tag{3.2}$$

where $X_*(x) = \sqrt{\sin^2(\pi l/\omega) - \sin^2(\pi x/\omega)}$. Substituting Eq. (3.2) into the left side of Eq. (3.1), taking into account Eqs. (1.2), and applying some transformations, we obtain a nonlinear integrodifferential equation with respect to the unknown function $q(x)$:

$$-\frac{1}{\pi X_*(x)}\Big(\int_{-l_0}^{l_0} \frac{X_*(t)q(t)(\pi/\omega)\cos(\pi t/\omega)}{\sin(\pi t/\omega) - \sin(\pi x/\omega)}\, dt + \int_{-l_0}^{l_0} \frac{X_*(t)f(t)(\pi/\omega)\cos(\pi t/\omega)}{\sin(\pi t/\omega) - \sin(\pi x/\omega)}\, dt\Big) = \frac{2\mu}{1+\varkappa_0}\frac{\partial}{\partial x}\,[C(x,q)\,q(x)]. \tag{3.3}$$

To solve the singular integrodifferential equation (3.3), we have to find the values of the concentrated forces P_{mn} ($m = \pm 1, \pm 2, \ldots$, $n = \pm 1, \pm 2, \ldots$). As the problem is symmetric and periodic, we obtain $P_{mn} = P_{m1}$ ($m = 1, 2, \ldots$). To determine P_{mn}, we use Hooke's law and the method of matching of two asymptotic curves of the sought solution. According to Hooke's law, the value of the concentrated force P_{mn} acting on each point of attachment from the side of the stringer is

$$P_{mn} = \frac{E_s F}{2y_0 m}\,\Delta v_{mn} \qquad (m = 1, 2, \ldots; \quad n = 1, 2, \ldots). \tag{3.4}$$

Here, E_s is Young's modulus of the stringer material, F is the cross-sectional area of the stringer, $2y_0 m$ is the distance between the attachment points, and Δv_{mn} is the relative displacement of the attachment points considered, which is equal to the elongation of the corresponding segment of the stringer.

Let us denote the radius of the attachment point (adhesion area) by a_0. We use the natural assumption that the relative elastic displacement of the points $z = m_* L + i(ny_0 - a_0)$ and $z = m_* L - i(ny_0 - a_0)$ in the considered problem of the elasticity theory is equal to the relative displacement Δv_{mn} of the attachment points [8]. This additional condition of compatibility of displacements allows us to find the solution of the problem posed above.

Using the complex potentials (2.1), (2.2), and (2.5) and the Kolosov–Muskhelishvili formula [15], we find the relative displacement of the attachment points Δv_{mn}:

$$\Delta v_{kr} = \Delta v_{kr}^{(0)} + \Delta v_{kr}^{(1)} + \Delta v_{kr}^{(2)}.$$

Here, we have

$$\Delta v_{kr}^{(0)} = \frac{1}{2\pi\mu(1+\varkappa_0)h} {\sum_{m=-\infty}^{\infty}}' {\sum_{n=-\infty}^{\infty}}' P_{mn}\Big(\varkappa_0 \ln\frac{C_1}{C_2} + \frac{2by_0C_3[2k(k-m_*)L^2 + C_3 a_0]}{C_2 C_1}\Big),$$

$$\Delta v_{kr}^{(1)} = \frac{\sigma_0}{\mu}\Big[\frac{1}{4}(3-\varkappa_0)\sin\frac{\pi C_3}{\omega} + \frac{1+\varkappa_0}{2\sqrt{2}}\sqrt{A-B}$$
$$- \frac{1}{\sqrt{2}\,A}\sin\frac{\pi C_3}{\omega}\Big(\sin\frac{\pi L}{\omega}\sqrt{A+B} + \sin\frac{\pi C_3}{\omega}\sqrt{A-B}\,\Big)\Big],$$

$$\Delta v_{kr}^{(2)} = \frac{1}{2\pi\mu}\int_{\lambda_0}^{l_0} f_1(t,l)q(t)\,dt - \frac{d_1}{\pi\mu}\int_{\lambda_0}^{l_0} X_*(t)f_2(t,l)q(t)\,dt$$
$$+ \frac{1}{2\pi\mu}\int_0^{l_0} f_1(t,l)f(t)\,dt - \frac{d_1}{\pi\mu}\int_0^{l_0} X_*(t)f_2(t,l)f(t)\,dt,$$

$$B = 1 - \sin^2(\pi l/\omega) - \sin^2(\pi C_3/\omega), \quad C_1 = (k-m_*)^2L^2 + a_0^2, \quad \lambda_0 = \sin(\pi\lambda/\omega),$$

$$C_2 = (k-m_*)^2L^2 + C_3^2, \quad C_3 = by_0 - a_0, \quad b = r - n, \quad d_1 = 2\sin(\pi C_3/\omega),$$

$$f_1(t,l) = \frac{1}{2}(1+\varkappa_0)C\ln\frac{D^2\cos^2\varphi + (D\sin\varphi - X_*(t))^2}{D^2\cos^2\varphi + (D\sin\varphi + X_*(t))^2},$$

$$f_2(t,l) = \frac{C}{D(d^2+d_1^2)}\Big[d\cos\varphi + d_1\sin\varphi + \frac{1}{2}d_1(d\sin\varphi - d_1\sin\varphi)\Big],$$

$$d = \sin^2(\pi t/\omega) - 1 + d_1, \qquad C = (\pi/\omega)\cos(\pi t/\omega),$$

$$D^2 = A = \sqrt{B^2 + d_1^2}, \qquad \varphi = (1/2)\arctan(d_1/B).$$

The sought values of the concentrated forces P_{mn} are determined by solving the infinite system of equations (3.4). System (3.4) and the singular integrodifferential equation (3.3) are related and should be solved together. The integrodifferential equation (3.3) is presented in the form

$$-\frac{1+\varkappa_0}{2\mu}\int_{-l_0}^{x} Q(x)\,dx = C(x,q)q(x), \qquad \lambda \leqslant |x - m\omega| \leqslant l, \tag{3.5}$$

where

$$Q(x) = \frac{1}{\pi X_*(x)}\int_{-l_0}^{l_0}\frac{X_*(t)[q(t)+f(t)](\pi/\omega)\cos(\pi t/\omega)}{\sin(\pi t/\omega) - \sin(\pi x/\omega)}\,dt.$$

We divide the segment $[-l_0, l_0]$ by the M nodal points t_m ($m = 1, 2, \ldots, M$) and require conditions (3.5) to be satisfied at the nodal points that belong to the tip zones. As a result, we obtain an algebraic system of M_1 equations (M_1 is the number of nodal points that belong to the tip zone of the crack) for determining the approximate values of $q(t_m)$ ($m = 1, 2, \ldots, M_1$):

$$\begin{gathered} C_0 Q(t_1) = C(t_1, q(t_1))q(t_1), \\ C_0(Q(t_1) + Q(t_2)) = C(t_2, q(t_2))q(t_2), \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ C_0\sum_{m=1}^{M_1} Q(t_m) = C(t_{M_1}, q(t_{M_1}))q(t_{M_1}). \end{gathered} \tag{3.6}$$

Here, we have $C_0 = -(1+\varkappa_0)\pi l_0/(2\mu M)$.

In constructing the algebraic systems, all intervals of integration were reduced to the interval $[-1, 1]$, and then the integrals were replaced by finite sums with the use of the Gauss quadrature formulas.

In the case of linearly elastic bonds, systems (3.4) and (3.6) are linear. To solve these systems numerically, we used the Gauss method with a choice of the element and calculated the stress intensity factors:

$$K_{\mathrm{I}} = K_{\mathrm{I}}^{\mathrm{load}} + K_{\mathrm{I}}^{\mathrm{stress}}.$$

Here, $K_{\mathrm{I}}^{\mathrm{load}}$ and $K_{\mathrm{I}}^{\mathrm{stress}}$ are the stress intensity factors induced by the action of the force load and stresses arising in the tip zone of the crack, respectively:

$$K_{\mathrm{I}}^{\mathrm{load}} = \frac{1}{\sqrt{\pi\sin(\pi l/\omega)}}\int_{-l_0}^{l_0} f(x)\sqrt{\frac{\sin(\pi x/\omega) + \sin(\pi l/\omega)}{\sin(\pi l/\omega) - \sin(\pi x/\omega)}}\,\frac{\pi}{\omega}\cos\frac{\pi x}{\omega}\,dx,$$

$$K_{\mathrm{I}}^{\mathrm{stress}} = \frac{1}{\sqrt{\pi\sin(\pi l/\omega)}}\int_{-l_0}^{l_0} q(x)\sqrt{\frac{\sin(\pi x/\omega) + \sin(\pi l/\omega)}{\sin(\pi l/\omega) - \sin(\pi x/\omega)}}\,\frac{\pi}{\omega}\cos\frac{\pi x}{\omega}\,dx.$$

The state of the critical equilibrium is described by the condition

$$G_b = G_n, \tag{3.7}$$

where

$$G_b = (1 - v)K_{\rm I}^2/(2\mu)$$

is the rate of strain energy release [7], and

$$G_n = h \int\limits_{l-\lambda}^{l} \frac{\partial}{\partial l}\,(v^+ - v^-)q(x)\,dx$$

is the rate of strain energy consumption by the bonds in the tip zone of the crack.

Condition (3.7) is a necessary but not a sufficient condition for determining the critical state of cracks with the tip zone. Therefore, an additional condition is necessary for determining the critical equilibrium state of the crack tip and the tip zone. As such a condition, we imply that the bonds on the face ($x_0 = \pm\lambda$) are broken if

$$v^+(x_0, 0) - v^-(x_0, 0) = \delta_{\rm cr}, \tag{3.8}$$

where $\delta_{\rm cr}$ is the critical tension of the bonds.

The joint solution of Eqs. (3.4) and (3.6)–(3.8) with a specified crack length and characteristics of the bonds allows us to find the critical external load and the size of the tip zone $d_{\rm cr} = l - \lambda$ in the case of the critical equilibrium state of the crack tip. The rate of strain energy consumption $G_{\rm cr}(d_{\rm cr}, l)$ found from this solution is the energy characteristic of resistance to fracture, i.e., $G_{\rm cr} = G_n(d_{\rm cr}, l)$. For specified sizes of the cracks and the tip zone, we can use the critical values $\delta_{\rm cr}$ and $G_{\rm cr}$ to identify the regimes of equilibrium and crack growth in a reinforced medium under monotonic loading.

If the conditions $G_b \geqslant G_{\rm cr}$ and $v^+ - v^- < \delta_{\rm cr}$ are satisfied for a specified size of the tip zone, the crack tip propagates, and the length of the tip zone increases, though the bonds are not broken. This stage of crack development can be considered as its adaptation to a specified level of external loading. The propagation of the crack tip is accompanied by breakdown of the bonds on the face of the tip zone if the conditions $G_b \geqslant G_{\rm cr}$ and $v^+ - v^- \geqslant \delta_{\rm cr}$ are satisfied.

If the inequalities $G_b < G_{\rm cr}$ and $v^+ - v^- \geqslant \delta_{\rm cr}$ are valid, the bonds are broken, but the crack tip does not propagate further; the end of the tip zone decreases and tends to a critical (for a given level of loading) value.

Finally, under the conditions $G_b < G_{\rm cr}$ and $v^+ - v^- < \delta_{\rm cr}$, the locations of the crack tip and the tip zone are not changed.

Thus, the magnitude of external loading and the critical parameters of the medium $G_{\rm cr}$ and $\delta_{\rm cr}$ determine the character of the fracture:

— propagation of the crack tips with simultaneous propagation of the tip zones;

— reduction of the sizes of the tip zones with no propagation of the periodic system of cracks;

— propagation of the crack tips with simultaneous breakdown of the bonds on the faces of the tip zones.

For a nonlinear law of bond deformation, the forces in the tip zones are found by an iterative algorithm similar to the method of elastic solutions [16].

For $v^+ - v^- \leqslant v_*$, the law of deformation of interparticle bonds (adhesion forces) is assumed to be linear.

The first step of the iterative process is solving system (3.4), (3.6) for linearly elastic interparticle bonds. Subsequent iterations are only performed if the relation $v^+ - v^- > v_*$ is valid for the tip zones. Such iterations are used to solve the system of equations in each approximation for quasi-elastic bonds with the effective compliance varying along the tip zones of the cracks and depending on the magnitude of forces in the bonds, which was obtained at the previous step of calculations.

The calculation of the effective compliance is similar to the calculation of the secant modulus in the method of variable parameters of elasticity [17]. The iterative process is terminated when the forces along the tip zones obtained at two consecutive iterations differ insignificantly.

The nonlinear segment of the bond deformation curve is described by a bilinear dependence whose uprising segment corresponds to elastic deformation of the bonds. For $v^+ - v^- > v_*$, the law of deformation is described by a nonlinear dependence determined by the points (v_*, σ_*) and $(\delta_{\rm cr}, \sigma_{\rm cr})$; in the case with $\sigma_{\rm cr} \geqslant \sigma_*$, there is an increasing linear dependence (linear hardening corresponding to elastoplastic deformation of the bonds).

Figure 2 shows the forces in the bonds of the tip zones of the cracks q/σ_0 as functions of the dimensionless coordinate x/l for the following values of the free parameters of the reinforced medium: $\nu = 0.3$, $\varepsilon_1 = a_0/L = 0.01$,

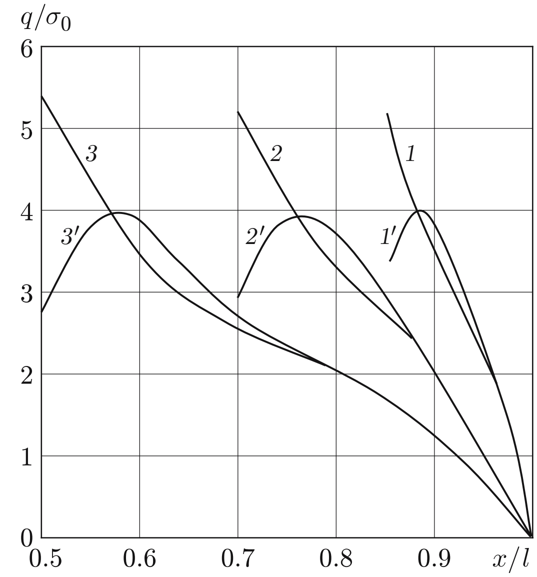

Fig. 2. Forces in the bonds between the crack faces in the tip zone q/σ_0 versus x/l: curves 1–3 and 1′–3′ refer to the linear and nonlinear law of deformation of the bonds, respectively; $d/l = 0.15$ (1 and 1′), 0.3 (2 and 2′), and 0.5 (3 and 3′).

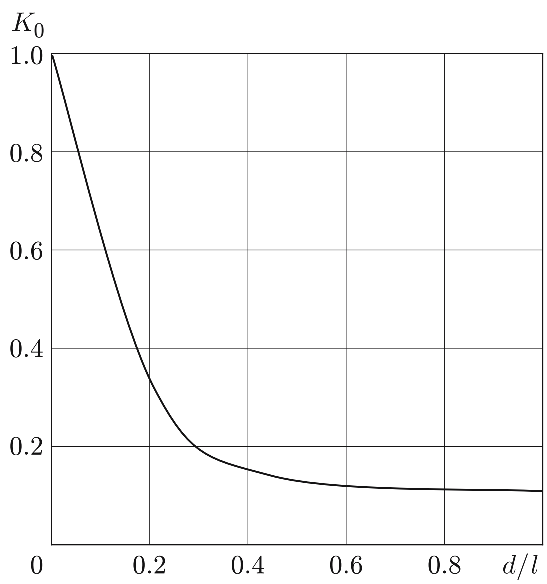

Fig. 3. Relative stress intensity factor K_0 versus the size of the tip zones of the cracks.

$\varepsilon = y_0/L = 0.25$, $E = 7.1 \cdot 10^4$ MPa (V95 alloy), $E_s = 11.5 \cdot 10^4$ MPa (aluminum–steel composite), $F/(y_0 h) = 1$, $v_* = 10^{-6}$ m, $\sigma_* = 130$ MPa, $\sigma_{\rm cr}/\sigma_* = 2$, $\delta_{\rm cr} = 2 \cdot 10^{-6}$ m, and effective compliance of the bonds $C = 1.9 \times 10^{-7}$ m/MPa. The number of stringers and attachment points was assumed to be 14 [18].

The calculations show that the presence of a regular system of stringers decreases the stress intensity factors, the forces in the bonds between the crack faces, and crack opening. For a linear law of deformation of the bonds, the forces reach the maximum values at the end of the tip zones. A similar dependence is observed for the crack opening values.

Figure 3 shows the relative stress intensity factor $K_0 = K_{\rm I}/K_{\rm I}^{\rm load}$ (which can be considered as a hardening coefficient) as a function of the size of the tip zones of the cracks. The calculations show that the hardening coefficient decreases with decreasing relative compliance.

An analysis of the critical equilibrium state of the reinforced plate with a periodic system of cracks with the bonds in the tip zones under tensile loading reduces to a parametric study of the solution of the algebraic systems (3.4), (3.6) with different laws of deformation of the bonds, sizes of the tip zones of the cracks, and elastic constants of the plate material. The forces in the bonds and the crack opening are determined directly from the solution of the algebraic systems in each approximation. The values of the stress intensity factors and the rates of energy release and consumption are calculated by the formulas given above.

The calculations show that the presence of reinforcing elements reduces the deformation of the stretched plate in the direction perpendicular to crack alignment, which leads to a substantial decrease in the stress intensity factor in the vicinity of the crack tips. At certain values of the geometric parameters, the stress intensity factors are negative, i.e., the crack faces are in contact with each other. Interaction of the crack faces induces contact stresses. In this case, the problem should be solved in a different formulation.

4. Partial Contact of the Crack Faces with the Bonds. We consider the case where the size of the tip zone with adhesion forces (bonds) is greater than the contact zone of the crack. Under external loading, the forces $q(x)$ arise between the crack faces. Correspondingly, the normal stresses $p(x)$ arise on the segments where the crack faces are in contact. The values of these stresses $q(x)$ and $p(x)$, and also the size of the contact zone are not known in advance and have to be determined in the course of solving the boundary-value problem of fracture mechanics.

In the problem considered, the boundary conditions have the form

$$\sigma_y - i\tau_{xy} = \begin{cases} 0, & y = 0, \quad |x - m\omega| < \lambda, \\ q(x), & y = 0, \quad \lambda \leqslant |x - m\omega| < \lambda_1, \\ p(x), & y = 0, \quad \lambda_1 \leqslant |x - m\omega| \leqslant l. \end{cases}$$

The constitutive relations of the problem posed should be supplemented by the equations

$$v^+(x) - v^-(x) = \begin{cases} 0, & y = 0, \quad \lambda_1 \leqslant |x - m\omega| \leqslant l, \\ C(x, q)q(x), & y = 0, \quad \lambda \leqslant |x - m\omega| < \lambda_1. \end{cases} \tag{4.1}$$

In the case considered, we obtain the following boundary-value problem for determining the analytical functions $\Phi_1(z)$ and $\Omega_1(z)$:

$$\Phi_1(x) + \overline{\Phi_1(x)} + \Omega_1(x) = \begin{cases} f(x), & y = 0, \quad |x - m\omega| < \lambda, \\ f(x) + q(x), & y = 0, \quad \lambda \leqslant |x - m\omega| < \lambda_1, \\ f(x) + p(x), & y = 0, \quad \lambda_1 \leqslant |x - m\omega| \leqslant l. \end{cases} \tag{4.2}$$

The solution of the boundary-value problem (4.2) should be found in the class of functions bounded everywhere. The sought solution of the problem is written as

$$\Omega_1(z) = 0, \qquad \Phi_1(z) = \frac{X(z)}{2\pi i} \int_{-l_0}^{l_0} \frac{F_*(x)(\pi/\omega)\cos(\pi x/\omega)}{\sin(\pi x/\omega) - \sin(\pi z/\omega)}\, dx.$$

Here, we have

$$F_*(x) = \begin{cases} f(x)/X(x), & |x - m\omega| < \lambda, \quad X(z) = \sqrt{\sin^2(\pi z/\omega) - \sin^2(\pi l/\omega)}, \\ (q(x) + f(x))/X(x), & \lambda \leqslant |x - m\omega| < \lambda_1, \\ (p(x) + f(x))/X(x), & \lambda_1 \leqslant |x - m\omega| \leqslant l. \end{cases}$$

Taking into account the behavior of the function $\Phi_1(z)$ at infinity, we write the condition of solvability of the boundary-value problem:

$$\int_{-l_0}^{l_0} \frac{[f(x) + q(x) + p(x)](\pi/\omega)\cos(\pi x/\omega)}{X_*(x)}\, dx = 0. \tag{4.3}$$

Relation (4.3) is used to determine the size of the tip contact zone.

Using the solution obtained above, we find

$$\frac{2\mu}{1+\varkappa_0}\,\frac{d}{dx}\,(v^+ - v^-) = -\frac{X_*(x)}{\pi}\int\limits_{-l_0}^{l_0}\frac{[f(x)+q(x)+p(x)](\pi/\omega)\cos(\pi t/\omega)}{X_*(x)(\sin(\pi t/\omega)-\sin(\pi x/\omega))}\,dt.$$

Taking into account conditions (4.1), we obtain a system of singular integral equations

$$\int\limits_{-l_0}^{l_0}\frac{[f(x)+q(x)+p(x)](\pi/\omega)\cos(\pi t/\omega)}{X_*(x)(\sin(\pi t/\omega)-\sin(\pi x/\omega))}\,dt = 0 \qquad (\lambda_1 \leqslant |x - m\omega| \leqslant l),$$

$$-\frac{X_*(x)}{\pi}\int\limits_{-l_0}^{l_0}\frac{[f(x)+q(x)+p(x)](\pi/\omega)\cos(\pi t/\omega)}{X_*(x)(\sin(\pi t/\omega)-\sin(\pi x/\omega))}\,dt = \frac{2\mu}{1+\varkappa_0}\,\frac{d}{dx}\,[C(x,q)q(x)] \tag{4.4}$$

$$(\lambda \leqslant |x - m\omega| < \lambda_1).$$

These equations are solved by a collocation scheme with approximation of unknown functions. In constructing the algebraic systems, all intervals of integration were reduced to the interval $[-1, 1]$, and then the integrals were replaced by finite sums with the use of the Gauss quadrature formulas. The derivatives in the right sides of the integrodifferential equations (4.4) were replaced by finite-difference approximations.

As the sizes of the contact zones are unknown, the resultant algebraic system is nonlinear even in the case of linear bonds. This system was solved by the method of consecutive approximations [17].

The sizes of the contact zone $(d_1)_* = d_1/L$ as functions of the crack length $l_* = l/L$ for $\varepsilon = 0.15$ and the same values of the free parameters of the reinforced medium as those used in Sec. 3 are given below. The resultant values are $(d_1)_* = 0.101$, 0.152, 0.206, 0.231, 0.259, and 0.335, which correspond to $l_* = 0.4$, 0.5, 0.6, 0.7, 0.8, and 0.9, respectively.

An analysis of the model of partial closing of the crack in an isotropic medium reinforced by a regular system of stringers reduces to a parametric study of the system of singular integral and integrodifferential equations (4.3) and (4.4) and an infinite algebraic system of the form (3.4) for different geometric and physical parameters of the medium reinforced by stringers, laws of deformation of the bonds, and sizes of the tip zones of the cracks. The contact stresses $p(x)$, the forces in the bonds $q(x)$, and the sizes of the contact zones are determined directly from the solution of the resultant algebraic systems.

Finally, it should be noted that the relations derived in the present work allow one to solve an inverse problem, i.e., determine the characteristics of medium reinforcement and the stress state of the plate, which ensure the contact of the crack faces in a zone determined in advance.

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